The amplitudes for the $K \to \pi\ell^+\ell^-$ decays: theoretical aspects

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based on: G. D'Ambrosio, D. Greynat, M.K., JHEP02, 049 (2019) [arXiv:1812.00735] and work in progress

OUTLINE

I. Introduction

II. Toward predicting the amplitude for $K^{\pm} \to \pi^{\pm} \ell^{+} \ell^{-}$

III. Summary - Conclusions

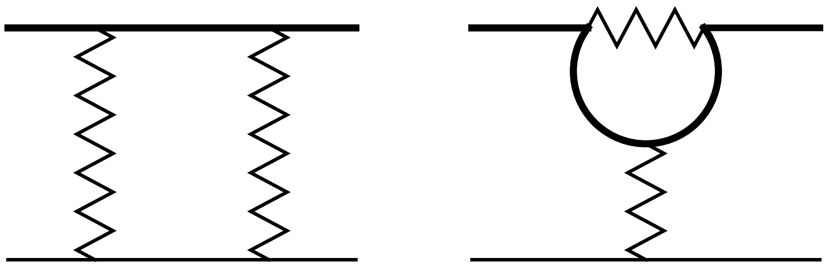
I. Introduction

Rare kaon decays proceed through FCNC, are suppressed in the SM
$\longrightarrow$ interesting window into new physics

For a review, see V. Cirigliano et al, Rev Mod Phys 84, 399 (2012)

Particularly interesting examples

$$K^+ \to \pi^+ \nu\bar{\nu} \text{ [NA62]} \qquad K_L \to \pi^0 \nu\bar{\nu} \text{ [KOTO]}$$



- dominated by short-distances
- clean SM prediction, hadronic matrix elements from $K_{\ell 3}$

F. Mescia, C. Smith, Phys. Rev. D 76, 034017 (2007)

$$Br(K^+ \to \pi^+ \nu\bar{\nu})_{\text{SM}} = 8.39(30) \cdot 10^{-11} \left[\frac{|V_{cb}|}{40.7 \cdot 10^{-3}}\right]^{2.8} \left[\frac{\gamma}{73.2^{\circ}}\right]^{0.74}$$

$$Br(K_L \to \pi^0 \nu\bar{\nu})_{\text{SM}} = 3.36(5) \cdot 10^{-11} \left[\frac{|V_{ub}|}{3.88 \cdot 10^{-3}}\right]^{2} \left[\frac{|V_{cb}|}{40.7 \cdot 10^{-3}}\right]^{2} \left[\frac{\sin\gamma}{\sin 73.2^{\circ}}\right]^{2}$$

J. Brod et al., Phys Rev D 83, 034030 (2011)

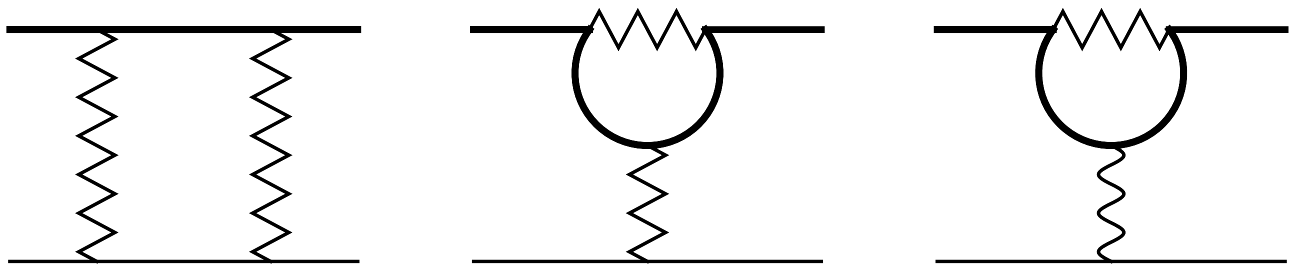
A. J. Buras et al, JHEP 1511, 33 (2011)

For a review, see A. Buras et al, Rev Mod Phys 80, 965 (2008)

For the CP conserving decays considered here

$$K^{\pm} \to \pi^{\pm}\gamma^{*} \to \pi^{\pm}\ell^{+}\ell^{-} \qquad\qquad K_S \to \pi^{0}\gamma^{*} \to \pi^{0}\ell^{+}\ell^{-}$$

the situation is less favourable



- similar short-distance parts as in $K \to \pi\nu\bar{\nu}$
- long distances dominate the amplitudes
- analogues, in the kaon sector, of $b \to s\ell^{+}\ell^{-}$ transitions
- any LFUV effect invoked in order to explain the anomalies seen at LHCb

might also manifest itself here A. Crivellin et al., Phys. Rev. D 93, 074038 (2016)

$$R_{K^{\pm}} \equiv \frac{\mathrm{Br}[K^{\pm} \to \pi^{\pm}\mu^{+}\mu^{-}]}{\mathrm{Br}[K^{\pm} \to \pi^{\pm}e^{+}e^{-}]} = \begin{cases} 0.313(71) & \text{[PDG average]} \\ 0.309(43) & \text{[NA48/2 alone]} \end{cases}$$

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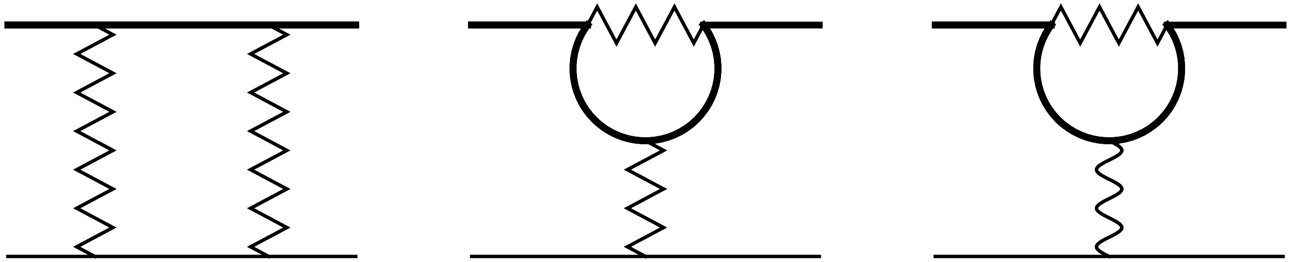

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What does the SM predict?

General structure of the amplitude

$$\mathcal{A}^{K\to\pi\ell^+\ell^-}(s) = -e^2 \times \bar{\mathrm{u}}(p_-)\gamma_\rho \mathrm{v}(p_+) \times \frac{1}{s} \times i\int d^4x\, \langle\pi(p)|T\{j^\rho(0)\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)\}|K(k)\rangle$$

$$-e^2 \times \bar{\mathrm{u}}(p_-)\gamma_\rho \mathrm{v}(p_+) \times \left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right) \times \frac{C_{7V}(\nu)}{4\pi\alpha}\, \langle\pi(p)|(\bar{s}\gamma^\rho d)(0)|K(k)\rangle$$

$$= -e^2 \times \bar{\mathrm{u}}(p_-)\gamma^\rho \mathrm{v}(p_+)(k+p)_\rho \times \frac{W_{K\pi}(z)}{16\pi^2 M_K^2} \qquad [z \equiv s/M_K^2]$$

$$W_{K\pi}(z) = W_{K\pi}^{\mathrm{LD}}(z;\nu) + W_{K\pi}^{\mathrm{SD}}(z;\nu)$$

$$\left[s(k+p)_\rho - (M_K^2 - M_\pi^2)(k-p)_\rho\right] \times \frac{W_{K\pi}^{\mathrm{LD}}(z;\nu)}{16\pi^2 M_K^2} = i\int d^4x\, \langle\pi(p)|T\{j_\rho(0)\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)\}|K(k)\rangle$$

$$\frac{W_{K\pi}^{\mathrm{SD}}(z;\nu)}{16\pi^2 M_K^2} = -\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right) \times \frac{C_{7V}(\nu)}{4\pi\alpha} \times C_{K\pi} f_+(s)$$

$$\langle\pi(p)|\,(\bar{s}\,\gamma_\rho d)(0)|K(k)\rangle = C_{K\pi}\left[(k+p)_\rho f_+^{K\pi}(s) + (k-p)_\rho f_-^{K\pi}(s)\right]$$

General structure of the amplitude

$$\begin{aligned}\mathcal{A}^{K\to\pi\ell^+\ell^-}(s) &= -e^2 \times \bar{\mathrm{u}}(p_-)\gamma_\rho \mathrm{v}(p_+) \times \frac{1}{s} \times i\int d^4x\, \langle \pi(p)| T\{j^\rho(0)\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)\}|K(k)\rangle \\ &\quad -e^2 \times \bar{\mathrm{u}}(p_-)\gamma_\rho \mathrm{v}(p_+) \times \left(-\frac{G_{\mathrm{F}}}{\sqrt{2}} V_{us}V_{ud}\right) \times \frac{C_{7V}(\nu)}{4\pi\alpha}\, \langle \pi(p)|(\bar{s}\gamma^\rho d)(0)|K(k)\rangle \\ &= -e^2 \times \bar{\mathrm{u}}(p_-)\gamma^\rho \mathrm{v}(p_+)(k+p)_\rho \times \frac{W_{K\pi}(z)}{16\pi^2 M_K^2} \qquad [z \equiv s/M_K^2]\end{aligned}$$

$$W_{K\pi}(z) = W_{K\pi}^{\mathrm{LD}}(z;\nu) + W_{K\pi}^{\mathrm{SD}}(z;\nu)$$

$$j^\rho(x) = \sum_{q=u,d,s} e_q (\bar{q}\gamma^\rho q)(x) \qquad \mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x) = \left(-\frac{G_{\mathrm{F}}}{\sqrt{2}} V_{us}V_{ud}\right) \times \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu)$$

$$\nu \frac{dC_{7V}(\nu)}{d\nu} = \frac{\alpha}{\alpha_s(\nu)} \sum_{J=1}^{6} \gamma_{J,7V}(\alpha_s) C_J(\nu)$$

E. Witten, Nucl. Phys. B 122, 109 (1977)
F. J. Gilman, M. B. Wise, Phys Rev D 21, 3150 (1980)
C. Dib et al., Phys Lett B 218, 487 (1989); Phys Rev D 39, 2639 (1989)
J. Flynn, L. Randall, Nucl Phys B 326, 31 (1989) [Nucl Phys B 334, 580 (1990)]
A. J. Buras et al., Nucl Phys B 423, 349 (1994)

Experimental situation

exp.	ref.	mode	number of events
BNL*	[1]	$K^+ \to \pi^+ e^+ e^-$	~ 500
BNL-E865*	[2]	$K^+ \to \pi^+ e^+ e^-$	$10\,300$
NA48/2*	[3]	$K^\pm \to \pi^\pm e^+ e^-$	$7\,263$
BNL-E787	[4]	$K^+ \to \pi^+ \mu^+ \mu^-$	~ 200
BNL-E865	[5]	$K^+ \to \pi^+ \mu^+ \mu^-$	~ 400
FNAL-E871	[6]	$K^\pm \to \pi^\pm \mu^+ \mu^-$	~ 100
NA48/2*	[7]	$K^\pm \to \pi^\pm \mu^+ \mu^-$	3120
NA48/1	[8]	$K_S \to \pi^0 e^+ e^-$	7
NA48/1	[9]	$K_S \to \pi^0 \mu^+ \mu^-$	6

[1] C. Alliegro *et al.*, Phys. Rev. Lett. **68**, 278 (1992)

[2] R. Appel *et al.* [E865 Collaboration], Phys. Rev. Lett. **83**, 4482 (1999) [hep-ex/9907045]

[3] J. R. Batley *et al.* [NA48/2 Collaboration], Phys. Lett. B **677**, 246 (2009) [arXiv:0903.3130 [hep-ex]]

[4] S. Adler *et al.* [E787 Collaboration], Phys. Rev. Lett. **79**, 4756 (1997) [hep-ex/9708012]

[5] H. Ma *et al.* [E865 Collaboration], Phys. Rev. Lett. **84**, 2580 (2000) [hep-ex/9910047]

[6] H. K. Park *et al.* [HyperCP Collaboration], Phys. Rev. Lett. **88**, 111801 (2002) [hep-ex/0110033]

[7] J. R. Batley *et al.* [NA48/2 Collaboration], Phys. Lett. B **697**, 107 (2011) [arXiv:1011.4817 [hep-ex]]

[8] J. R. Batley *et al.* [NA48/1 Collaboration], Phys. Lett. B **576**, 43 (2003) [hep-ex/0309075]

[9]J. R. Batley *et al.* [NA48/1 Collaboration], Phys. Lett. B **599**, 197 (2004) [hep-ex/0409011]

*: decay distribution

Experimental situation

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Differential decay rate

$$\frac{d\Gamma}{dz} = \frac{\alpha^2 M_K}{3(4\pi)^5}\lambda^{3/2}(1, z, M_\pi^2/M_K^2)\sqrt{1 - 4\frac{m_\ell^2}{zM_K^2}}\left(1 + 2\frac{m_\ell^2}{zM_K^2}\right)|W_{K\pi}(z)|^2 \quad z \equiv s/M_K^2$$

II. Predicting the amplitude for $K^{\pm} \rightarrow \pi^{\pm}\ell^{+}\ell^{-}$

Traditional approach: chiral perturbation theory

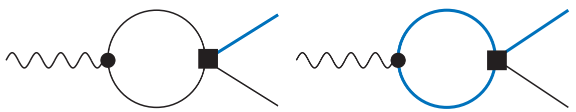
One loop

$$W_{+,S;1\mathrm{L}}(z) = G_F M_K^2 a_{+,S}^{\text{CT-1L}} + \frac{8\pi^2}{3}\frac{M_K^2}{M_\pi^2}\Bigg\{\alpha_{+,S}^{\text{tree}}\left[\frac{z-4\frac{M_\pi^2}{M_K^2}}{z}\bar{J}_{\pi\pi}(zM_K^2) - \frac{1}{48\pi^2}\right] + \tilde{\alpha}_{+,S}^{\text{tree}}\left[\frac{z-4}{z}\bar{J}_{KK}(zM_K^2) - \frac{1}{48\pi^2}\right]\Bigg\}$$

$$= G_F M_K^2 (a_{+,S}^{\text{1L}} + b_{+,S}^{\text{1L}} z) + \frac{8\pi^2}{3}\frac{M_K^2}{M_\pi^2}\alpha_{+,S}^{\text{tree}}\left[\frac{z-4\frac{M_\pi^2}{M_K^2}}{z}\bar{J}_{\pi\pi}(zM_K^2) + \frac{1}{24\pi^2}\right]$$

G. Ecker, A. Pich, E. de Rafael, Nucl Phys B 291, 692 (1987)

B. Ananthanarayan, I. S. Imsong, J. Phys. G 39, 095002 (2012)



$$\alpha_+^{\text{tree}} = \tilde{\alpha}_+^{\text{tree}} = \left(-\frac{G_F}{\sqrt{2}}V_{us}^* V_{ud}\right) M_\pi^2 \left(g_8 - \frac{13}{3}g_{27}\right) = -0.36 M_\pi^2 G_F = -8.16\cdot 10^{-8}$$

$$\alpha_S^{\text{tree}} = \left(-\frac{G_F}{\sqrt{2}}V_{us}^* V_{ud}\right) M_\pi^2 \left(\frac{5}{3}g_{27}\frac{3M_K^2 - 2M_\pi^2}{M_K^2 - M_\pi^2}\right) = -0.24 M_\pi^2 G_F = -5.36\cdot 10^{-8}$$

$$\tilde{\alpha}_S^{\text{tree}} = \left(-\frac{G_F}{\sqrt{2}}V_{us}^* V_{ud}\right) M_\pi^2 \left(-2g_8 + \frac{g_{27}}{3}\frac{M_K^2 - 6M_\pi^2}{M_K^2 - M_\pi^2}\right) = +1.11 M_\pi^2 G_F = +25.15\cdot 10^{-8}$$

$$a_+^{\text{CT-1L}} = \left(-\frac{1}{\sqrt{2}}V_{us}^* V_{ud}\right)\left(g_8 w_+^{(8)} - \frac{13}{3}g_{27}w_+^{(27)}\right)$$

Expanding the kaon loop to the term linear in z gives

$$a_{+,S}^{\text{1L}} = a_{+,S}^{\text{CT-1L}} + a_{+,S}^{\text{1L};\pi\pi} + a_{+,S}^{\text{1L};\bar{K}K},\ a_{+,S}^{\text{1L};\pi\pi} = -\frac{\alpha_{+,S}^{\text{tree}}}{6M_\pi^2 G_F},\ a_{+,S}^{\text{1L};\bar{K}K} = -\frac{\tilde{\alpha}_{+,S}^{\text{tree}}}{6M_\pi^2 G_F};\quad b_{+,S}^{\text{1L}} = b_{+,S}^{\text{1L};\bar{K}K} = \frac{\tilde{\alpha}_{+,S}^{\text{tree}}}{60M_\pi^2 G_F}$$

Traditional approach: chiral perturbation theory

One loop

$$W_{+,S;1\mathrm{L}}(z) = G_F M_K^2 a_{+,S}^{\text{CT-1L}} + \frac{8\pi^2}{3}\frac{M_K^2}{M_\pi^2}\Bigg\{\alpha_{+,S}^{\mathrm{tree}}\left[\frac{z-4\frac{M_\pi^2}{M_K^2}}{z}\bar{J}_{\pi\pi}(zM_K^2) - \frac{1}{48\pi^2}\right] + \tilde{\alpha}_{+,S}^{\mathrm{tree}}\left[\frac{z-4}{z}\bar{J}_{KK}(zM_K^2) - \frac{1}{48\pi^2}\right]\Bigg\}$$

$$= G_F M_K^2 (a_{+,S}^{\text{1L}} + b_{+,S}^{\text{1L}} z) + \frac{8\pi^2}{3}\frac{M_K^2}{M_\pi^2}\alpha_{+,S}^{\mathrm{tree}}\left[\frac{z-4\frac{M_\pi^2}{M_K^2}}{z}\bar{J}_{\pi\pi}(zM_K^2) + \frac{1}{24\pi^2}\right]$$

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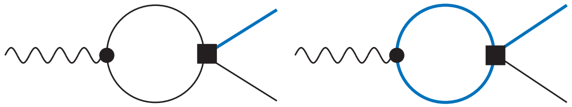


- Counterterms in $a_{+,S}^{1\mathrm{L}}$ not known (no predictivity)
- Substantial differences in the structures of $W_{+;1\mathrm{L}}(z)$ and $W_{S;1\mathrm{L}}(z)$, and therefore between a_+^{1L} and a_S^{1L}
- Pion loops are suppressed by the $\Delta I = 1/2$ rule in $W_{S;1\mathrm{L}}(z)$, but kaon loops are about twice as important as in $W_{+;1\mathrm{L}}(z)$
- Kaon loops contribute substantially to $a_{+,S}^{\text{1L}}$: $a_+^{\text{1L};\bar{K}K} = +0.06$, $a_S^{\text{1L};\bar{K}K} = -0.19$ (these contributions are proportional to $\tilde{\alpha}_+^{\mathrm{tree}}$ and $\tilde{\alpha}_S^{\mathrm{tree}}$, respectively)
- Higher-order corrections will move $\tilde{\alpha}_{+,S}^{\mathrm{tree}}$ to their phenomenological values $\tilde{\alpha}_{+,S}$, which are however not known. Assuming, for the sake of illustration, that this change is of about the same size as the change from $\alpha_{+,S}^{\mathrm{tree}}$ to $\alpha_{+,S}$, i.e. $\tilde{\alpha}_+ \sim 2.5\tilde{\alpha}_+^{\mathrm{tree}}$, $\tilde{\alpha}_S \sim 1.3\tilde{\alpha}_S^{\mathrm{tree}}$ we would conclude that the kaon loops could contribute to a_+ (a_S) at the level of $\sim +0.15$ (~ -0.25) (argument is at best indicative, requires an explicit two-loop computation)
- Kaon loops provide only tiny slopes $b_{+,S}^{\text{1L}} \longrightarrow W_{+,S;1\mathrm{L}}$ gives a poor description of the data

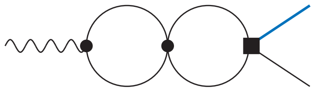
Traditional approach: chiral perturbation theory

Beyond one loop

$$W_{+,S;\mathrm{b1L}}(z) = G_F M_K^2 (a_{+,S} + b_{+,S} z)$$

G. D'Ambrosio et al., JHEP 9808, 004 (1998)

$$+ \frac{8\pi^2}{3} \frac{M_K^2}{M_\pi^2} \left[\alpha_{+,S} + \beta_{+,S} \frac{M_K^2}{M_\pi^2} (z - z_0) \right] \left(1 + \frac{M_K^2}{M_V^2} z \right) \left[\frac{z - 4\frac{M_\pi^2}{M_K^2}}{z} \bar{J}_{\pi\pi}(z M_K^2) + \frac{1}{24\pi^2} \right]$$



- Only loops with pions, loops with kaons included in the polynomial part
- Still not a complete two-loop representation, but neglected two-loop effects are small

G. D'Ambrosio, D. Greynat, M.K., JHEP02, 049 (2019)

- $\alpha_+, \beta_+ \ [\alpha_S, \beta_S]$ from slope and curvature of $K^+ \to \pi^+\pi^+\pi^- \ [K_S \to \pi^0\pi^+\pi^-]$ amplitude

$$\alpha_+ = -20.84(74)\cdot 10^{-8}, \quad \beta_+ = -2.88(1.08)\cdot 10^{-8}, \quad \alpha_S = -6.81(74)\cdot 10^{-8}, \quad \beta_S = -1.5(1.1)\cdot 10^{-8}$$

J. Bijnens et al. Nucl. Phys. B 648, 317 (2003)

- Gives a good fit to data

$$a_+ = -0.593(9)_{\mathrm{stat}}(1)_{\alpha_+}(6)_{\beta_+} \quad b_+ = -0.675(40)_{\mathrm{stat}}(16)_{\alpha_+}(2)_{\beta_+} \quad [\chi^2/\mathrm{dof} = 45.7/39]$$

$$(a_+ = -0.561(13) \quad b_+ = -0.694(40) \quad \beta_+ = 2.20(1.92)\cdot 10^{-8} \ [\chi^2/\mathrm{dof} = 38.3/38])$$

- Still no prediction for $a_{+,S}$ and $b_{+,S}$

$$G_F M_K^2 a_{+,S} = W_{+,S;\mathrm{b1L}}(0), \; G_F M_K^2 b_{+,S} = W'_{+,S;\mathrm{b1L}}(0) - \frac{1}{60}\left(\frac{M_K^2}{M_\pi^2}\right)^2 \left(\alpha_{+,S} - \beta_{+,S} \frac{s_0}{M_K^2} \right)$$

Predictivity: going beyond chiral perturbation theory

- Chiral perturbation theory and large-N_c

S. Friot et al., Phys Lett B 595, 301 (2004)

E. Coluccio Leskov et al., Phys Rev D 93, 094031 (2016)

- Lattice QCD

G. Isidori et al., Phys Lett B 633, 75 (2006)

N. H. Christ et al, Phys Rev D 92, 094512 (2015); D 94, 114516 (2016)

$\longrightarrow$ talks by A. Portelli and F. Ó hÓgáin

- $\cdots$

J. Portolés, J Phys Conf Series 800, 012030 (2017)

Predictivity: going beyond chiral perturbation theory

Rewrite $W_{+;2\mathrm{L}}(z)$ as the dispersive representation

$$W_{+;2\mathrm{L}}(z) = G_F M_K^2(a_+ + b_+ z) + \frac{z^2 M_K^4}{\pi}\int_0^\infty \frac{dx}{x^2}\frac{\mathrm{Abs}\,W_{+;2\mathrm{L}}(x/M_K^2)|_{\pi\pi}}{x - zM_K^2 - i0}$$

with

$$\frac{\mathrm{Abs}\,W_{+;2\mathrm{L}}(s/M_K^2)|_{\pi\pi}}{16\pi^2 M_K^2} = \theta(s-4M_\pi^2)\times\frac{s-4M_\pi^2}{s}\lambda_{K\pi}^{-1/2}(s)\Big[\mathrm{Re}\,F_V^\pi(s)|_{\mathcal{O}(E^4)}\times f_1^{\pi^+\pi^-\to K^+\pi^-}(s)|_{\mathcal{O}(E^2)} + F_V^\pi(s)|_{\mathcal{O}(E^2)}\times[\mathrm{Disp}\,f_1^{\pi^+\pi^-\to K^+\pi^-}(s)|_{\mathcal{O}(E^4)} - f_1^{\pi^+\pi^-\to K^+\pi^-}(s)|_{\mathcal{O}(E^2)}]\Big]$$

a_+ and b_+ are subtraction constants

Describe the contribution from $\pi\pi$ intermediate state by an unsubtracted DR

$$W_+|_{\pi\pi}(z) = \frac{1}{\pi}\int_0^\infty dx\frac{\mathrm{Abs}\,W_+(x/M_K^2)|_{\pi\pi}}{x-zM_K^2-i0}$$

with

$$\frac{\mathrm{Abs}\,W_+(s/M_K^2)|_{\pi\pi}}{16\pi^2M_K^2} = \theta(s-4M_\pi^2)\times\frac{s-4M_\pi^2}{s}\lambda_{K\pi}^{-1/2}(s)F_V^{\pi*}(s)\times f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$$

Predictivity: going beyond chiral perturbation theory

Then a_+ and b_+ are given by spectral sum rules

$$G_F M_K^2 a_+|_{\pi\pi} = W_+(0)|_{\pi\pi} = \frac{1}{\pi}\int_0^\infty \frac{dx}{x}\,\mathrm{Abs}\,W(x/M_K^2)|_{\pi\pi}$$

and

$$\begin{aligned} G_F M_K^2 b_+|_{\pi\pi} &= W'(0)|_{\pi\pi} - \frac{1}{60}\left(\frac{M_K^2}{M_\pi^2}\right)^2\left(\alpha_+ - \beta_+\frac{s_0}{M_\pi^2}\right) \\ &= \frac{M_K^2}{\pi}\int_0^\infty \frac{dx}{x^2}\,\mathrm{Abs}\,W(x/M_K^2)|_{\pi\pi} - \frac{1}{60}\left(\frac{M_K^2}{M_\pi^2}\right)^2\left(\alpha_+ - \beta_+\frac{s_0}{M_\pi^2}\right) \end{aligned}$$

requires $F_V^{\pi *}(s)$ and $f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$ beyond low-energy expansion

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requires $F_V^{\pi*}(s)$ and $f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$ beyond low-energy expansion

What about the other intermediate states?

How does one connect to the QCD continum? (scale dependence)

Predictivity: going beyond chiral perturbation theory

Proposal:

$$W_+(z) = W_+^{\pi\pi}(z) + W_+^{\rm res}(z;\nu) + W_+^{\rm SD}(z;\nu)$$

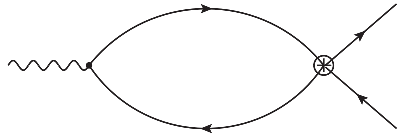
- $W_+^{\pi\pi}(z)$: contribution from (resonant) two-pion state

- $W_+^{\rm SD}(z)$: contributions from short distances

- $W_+^{\rm res}(z)$: contribution from intermediate energy range
  - described by a set of zero-width resonances
  - has to match the $\sim \ln(-s/\nu^2)$ behaviour at short distances

Predictivity: going beyond chiral perturbation theory

Short-distance behaviour



$$\lim_{q\to\infty} i\int d^4x\, e^{iq\cdot x} T\{j^\mu(x)\mathcal{L}_{\Delta S=1}(0)\} =$$

$$= \left(-\frac{G_{\rm F}}{\sqrt{2}}V_{us}^* V_{ud}\right)[q^\mu q^\rho - q^2\eta^{\mu\rho}] \times \bar{s}\gamma_\rho(1-\gamma_5)d \times \frac{1}{4\pi}\sum_{I=1}^{6} C_I(\nu)\,\xi_I(\alpha_s\,;\nu^2/q^2) + \mathcal{O}(q)$$

$$\xi_I(\alpha_s\,;\nu^2/q^2) = \sum_{p\geq 0}\sum_{r=0}^{p+1}\xi_{pr}^I\alpha_s^p(\nu)\ln^r(-\nu^2/q^2)$$

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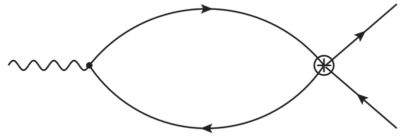

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$$\xi_I(\alpha_s\,;\nu^2/q^2) = \sum_{p\geq 0}\sum_{r=0}^{p+1} \xi^I_{pr}\alpha_s^p(\nu)\ln^r(-\nu^2/q^2)$$

One loop:

$$\xi^1_{01} = \frac{1}{4\pi}\,\frac{8}{9}\,N_c, \qquad \xi^2_{01} = \frac{1}{4\pi}\,\frac{8}{9}, \qquad \xi^1_{00} = \frac{1}{4\pi} \times \begin{cases} \frac{16}{27}\,N_c \ \text{NDR} \\ \frac{40}{27}\,N_c \ \text{HV} \end{cases}, \qquad \xi^2_{00} = \frac{1}{4\pi} \times \begin{cases} \frac{16}{27} \ \text{NDR} \\ \frac{40}{27} \ \text{HV} \end{cases},$$

$$\xi^3_{01} = -\frac{1}{4\pi}\,\frac{8}{9}, \qquad \xi^4_{01} = -\frac{1}{4\pi}\,\frac{8}{9}\,N_c, \qquad \xi^3_{00} = \frac{1}{4\pi} \times \begin{cases} -\frac{16}{27} \ \text{NDR} \\ -\frac{40}{27} \ \text{HV} \end{cases}, \qquad \xi^4_{00} = \frac{1}{4\pi} \times \begin{cases} -\frac{16}{27}\,N_c \ \text{NDR} \\ -\frac{40}{27}\,N_c \ \text{HV} \end{cases}$$

$$\xi^{5,6}_{00} = \xi^{5,6}_{01} = 0$$

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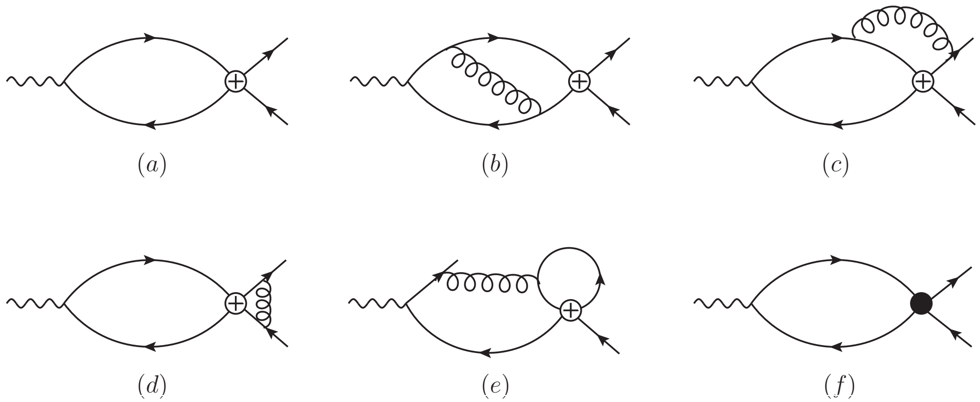
Short-distance behaviour



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Two loops:

$$\nu\frac{d\xi_I(\alpha_s\,;\nu^2/s)}{d\nu} + \sum_{J=1}^{6}\gamma_{I,J}(\alpha_s)\xi_J(\alpha_s\,;\nu^2/s) = -\frac{\gamma_{I,7V}(\alpha_s)}{\alpha_s(\nu)}$$

$$\xi^I_{12} = \frac{1}{(4\pi)^2}\,\frac{4}{27}\left(N_c - \frac{1}{N_c}\right) \times (0\,,\,-8\,,\,+11\,,\,N_f\,,\,0\,,\,N_f)$$

$$\xi^I_{11} = \frac{1}{(4\pi)^2}\,\frac{8}{3}\left(N_c - \frac{1}{N_c}\right) \times \begin{cases} \left(\frac{N_c}{2}\,,\,-\frac{19}{18}\,,\,\frac{17}{9}\,,\,\frac{7}{6}-\frac{N_c}{2}\,,\,0\,,\,\frac{7}{6}\right) & \text{NDR} \\ \left(\frac{N_c}{2}\,,\,-\frac{5}{18}\,,\,\frac{13}{9}\,,\,\frac{7}{6}-\frac{N_c}{2}\,,\,0\,,\,\frac{7}{6}\right) & \text{HV} \end{cases}$$

G. Altarelli et al., Nucl. Phys. B **187**, 461 (1981)
A. J. Buras and P. H. Weisz, Nucl. Phys. B **333**, 66 (1990)
A. J. Buras et al., Nucl. Phys. B **370**, 69 (1992)
A. J. Buras et al., Nucl. Phys. B **400**, 37 (1993)
M. Ciuchini et al., Nucl. Phys. B **415**, 403 (1994)

$$\xi_{10} = ?$$

Predictivity: going beyond chiral perturbation theory

Two-pion state

Simple approach: unitarize both F_V^π and $f_1^{K^+\pi^-\to\pi^+\pi^-}$ using the inverse amplitude method

$$F_V^\pi(s)|_{\mathrm{IAM}} = \frac{1}{1 - \frac{s}{M_V^2} - \frac{\beta}{6F_\pi^2}(s - 4M_\pi^2)\,\bar{J}_{\pi\pi}(s)}$$

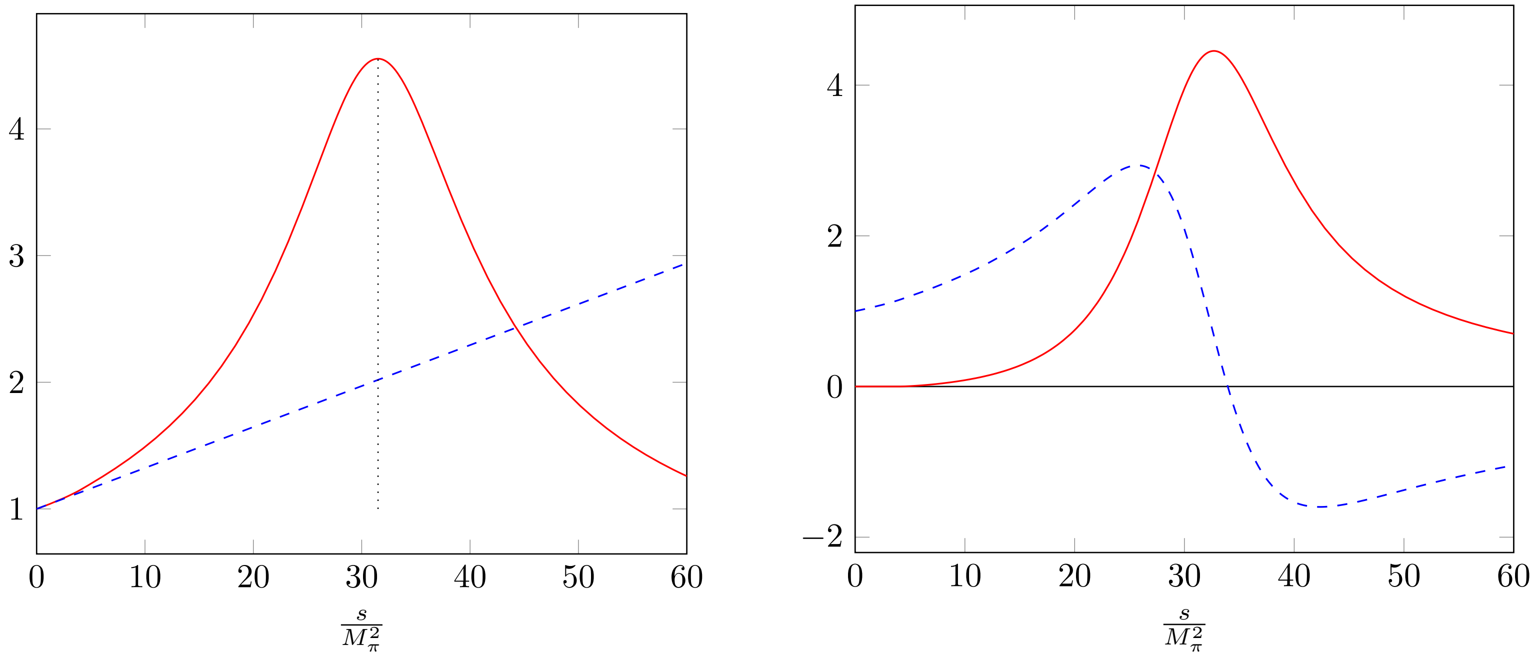
T. N. Truong, Phys Rev Lett 61, 2526 (1988)

A. Dobado et al, Phys Lett B 235, 134 (1990)

T. Hannah, Phys Rev D 55, 5613 (1997)

A. Dobado, J. R. Pelaez, Phys Rev D 56, 3057 (1997)

J. Nieves et al., Phys Rev D 65, 036002 (2002)



left: $|F_V^\pi(s)|$; right: $\mathrm{Re}\, F_V^\pi(s)$ (dashed) and $\mathrm{Im}\, F_V^\pi(s)$ (solid)

Predictivity: going beyond chiral perturbation theory

Two-pion state

Simple approach: unitarize both F_V^π and $f_1^{K^+\pi^-\to\pi^+\pi^-}$ using the inverse amplitude method

$$f_1^{K^+\pi^-\to\pi^+\pi^-}(s) = \frac{\frac{\alpha_+}{96\pi M_\pi^2} \times \lambda_{K\pi}^{1/2}(s)\sqrt{1-\frac{4M_\pi^2}{s}}}{1 - \frac{\beta_+}{\alpha_+}\frac{s-s_0}{M_\pi^2} - \frac{\beta}{6}\frac{s-4M_\pi^2}{F_\pi^2}\left[\bar{J}_{\pi\pi}(s) - \mathrm{Re}\,\bar{J}_{\pi\pi}(s_0)\right] + \frac{\beta}{6}\frac{s_0-4M_\pi^2}{F_\pi^2}(s-s_0)\mathrm{Re}\,\bar{J}'_{\pi\pi}(s_0)}$$

T. N. Truong, Phys Rev Lett 61, 2526 (1988)

A. Dobado et al, Phys Lett B 235, 134 (1990)

T. Hannah, Phys Rev D 55, 5613 (1997)

A. Dobado, J. R. Pelaez, Phys Rev D 56, 3057 (1997)

J. Nieves et al., Phys Rev D 65, 036002 (2002)

$$a_+|_{\pi\pi} = -1.58 \qquad b_+|_{\pi\pi} = -0.76 \qquad \text{for } \beta_+ = -0.85\cdot 10^{-8}$$

remark 1: $b_+ \sim a_+(M_K^2/M_\rho^2) \longrightarrow$ non VMD contributions to a_+ expected

G. D'Ambrosio et al., JHEP 9808, 004 (1998)

remark 2: position of the ρ resonance in $f_1^{\pi^+\pi^-\to K^+\pi^-}(s)$ much too low for $\beta_+ = -2.88(1.08)\cdot 10^{-8}$... (phase goes through $\pi/2$ at $s \sim M_\rho^2/2$!)

Predictivity: going beyond chiral perturbation theory

Two-pion state

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T. N. Truong, Phys Rev Lett 61, 2526 (1988)

A. Dobado et al, Phys Lett B 235, 134 (1990)

T. Hannah, Phys Rev D 55, 5613 (1997)

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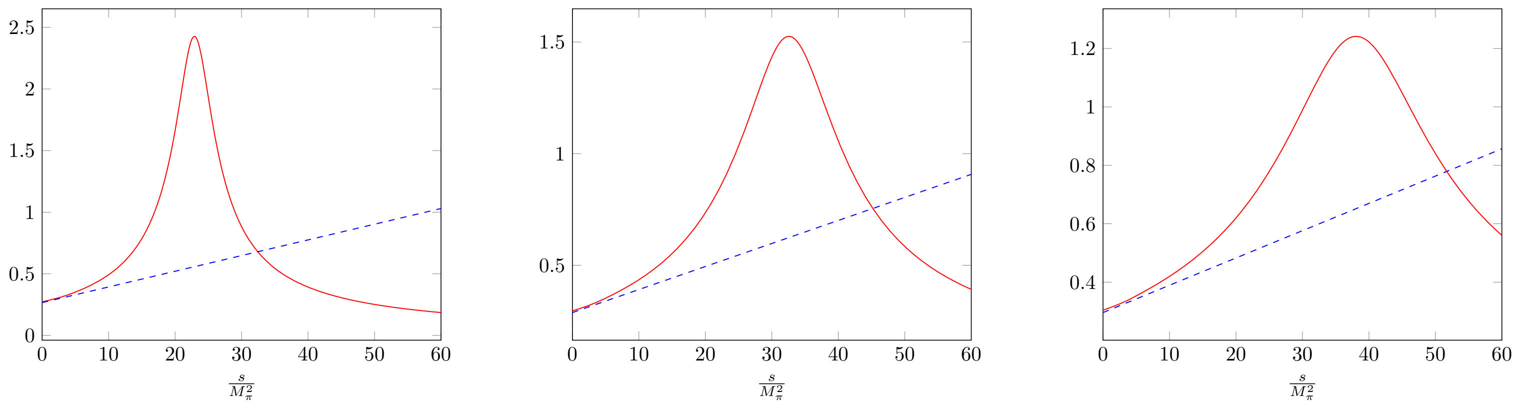
J. Nieves et al., Phys Rev D 65, 036002 (2002)



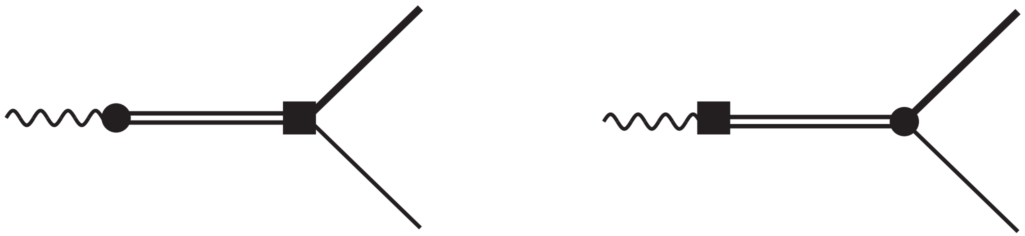
$|f_1^{\pi^+\pi^-\to K^+\pi^-}(s)/\lambda_{K\pi}^{1/2}(s)\sqrt{1-\frac{4M_\pi^2}{s}}|$ for $\alpha_+ = -20.84\cdot 10^{-8}$
left: $\beta_+ = -1.26\cdot 10^{-8}$; centre: $\beta_+ = -0.85\cdot 10^{-8}$; right: $\beta_+ = -0.72\cdot 10^{-8}$

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Other intermediate states?

Simplified proposal: modelled by an infinite tower of zero-width resonances, with couplings adjusted such as to match the short-distance behaviour

$$W_+(z) = W_+^{\pi\pi}(z) + W_+^{\rm res}(z;\nu) + W_+^{\rm SD}(z;\nu)$$



Possible resonances:
$J^{PC} = 1^{--}$, $I = 1$, $S = 0$, like $\phi(1020)$ [not $\rho(770)$!]
$J^{P} = 1^{-}$, $I = 1/2$, $S = \pm 1$, like $K^*(892)$

$$W_+^{\rm res}(z;\nu) = \sum_{V=\phi\cdots} \frac{f_V \widetilde{g}_V}{s - M_V^2 + i0} + \sum_{V=K^*\cdots} \frac{g_V \widetilde{f}_V}{s - M_V^2 + i0}$$

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Other intermediate states? Matching to short-distances?

Look for

$$W_+^{\rm res}(z;D) = \frac{f_+^{K^\pm\pi^\mp}(zM_K^2)}{4\pi}\int dx\,\frac{\rho_{\rm res}(x;D)}{x - zM_K^2 - i0}$$

$$\rho_{\rm res}(s;D) = A(D)(4\pi)^{2-\frac{D}{2}}\left(\frac{M^2}{\nu_{\rm MS}^2}\right)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\sum_{n\geq 1} M^2\mu_n(D)\delta(s-nM^2) \quad A(D) = A + A'(D-4) + \cdots$$

$$\int dx\,\frac{\rho_{\rm res}(x;D)}{x+wM^2} = A(D)(4\pi)^{2-\frac{d}{2}}\left(\frac{M^2}{\nu_{\rm MS}^2}\right)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\sum_{n\geq 1}\frac{\mu_n(D)}{(n+w)} \quad w \equiv -s/M^2 \quad M \sim 1{\rm GeV}$$

with the weights $\mu_n(D)$ satisfying

- $\sum_{n\geq 1}\frac{\mu_n(D)}{n} = 1 + \mathcal{O}(D-4)$
- $\mu_n(D) = (D-4)\bar{\mu}_n + \mathcal{O}\left((D-4)^2\right)$ as $D \to 4$
- $\xi(w) \equiv \sum_{n\geq 1}\frac{\bar{\mu}_n}{n(n+w)}$ converges
- $\xi(w) \sim \ln w$ as $w \to +\infty$

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Other intermediate states? Matching to short-distances?

At order $\mathcal{O}(\alpha_s^0)$, a solution (not unique!) can be found in the form

$$\mu_n(D) = \frac{n^{\frac{D-4}{2}}}{\zeta\left(3-\frac{D}{2}\right)+f(D)} \qquad f(D) = f(4) + (D-4)f'(4) + \cdots$$

$$W_+^{\rm res}(z;\nu) = \frac{f_+^{K^\pm\pi^\mp}(zM_K^2)}{4\pi} \times 16\pi^2 M_K^2 \left(\frac{G_{\rm F}}{\sqrt{2}} V_{us}^* V_{ud}\right) \sum_I C_I(\nu) \left\{\xi_{00}^I - \xi_{01}^I \left[\ln\frac{M^2}{\nu^2} + \psi\left(1 - z\frac{M_K^2}{M^2}\right)\right]\right\}$$

$$\psi\left(1 - z\frac{M_K^2}{M^2}\right) = -\gamma_E - zM_K^2 \int \frac{dx}{x}\, \frac{1}{x - zM_K^2} \sum_{n\geq 1} M^2 \delta(x - nM^2) \qquad \psi\left(1 - z\frac{M_K^2}{M^2}\right) \underset{z\to-\infty}{\sim} \ln\left(-z\frac{M_K^2}{M^2}\right)$$

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Other intermediate states? Matching to short-distances?

The construction can be extended to order $\mathcal{O}(\alpha_s)$

G. D'Ambrosio, D. Greynat, M. K., to appear

$$W_{K\pi}^{\rm res}(z;\nu) = -16\pi^2 M_K^2 \left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud}\right) \frac{C_{K\pi} f_+^{K\pi}(zM_K^2)}{4\pi}$$
$$\times \sum_{I=1}^{6} C_I(\nu) \Bigg\{ \xi_{00}^I + \alpha_s(\nu)\xi_{10}^I + \ln\frac{\nu^2}{M^2}\left[\xi_{01}^I + \alpha_s(\nu)\left(\xi_{11}^I + \xi_{12}^I \ln\frac{\nu^2}{M^2}\right)\right]$$
$$-\psi\left(1 - z\,\frac{M_K^2}{M^2}\right)\left[\xi_{01}^I + \alpha_s(\nu)\left(\xi_{11}^I + 2\xi_{12}^I \ln\frac{\nu^2}{M^2}\right)\right] + \alpha_s(\nu)\,\xi_{12}^I\left(\widetilde{\psi}\left(-z\,\frac{M_K^2}{M^2}\right) - \frac{\pi^2}{3} - 2\gamma_1\right)\Bigg\}$$

$$\widetilde{\psi}\left(-z\,\frac{M_K^2}{M^2}\right) = -2zM_K^2 \int \frac{dx}{x}\,\frac{1}{x - zM_K^2} \sum_{n\geq 1} M^2 (\ln n)\delta(x - nM^2)$$

$$\widetilde{\psi}\left(-z\,\frac{M_K^2}{M^2}\right) \underset{z\to-\infty}{\sim} \ln^2\left(-z\frac{M_K^2}{M^2}\right) + 2\left(\frac{\pi^2}{6} + \gamma_1\right) + \cdots$$

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Putting everything together

$$
\begin{aligned}
a_+ &= \int_0^\infty \frac{dx}{x}\,\frac{\rho_+^{\pi\pi}(x)}{G_F M_K^2} + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \left\{ \frac{C_{7V}(\nu)}{\alpha} + \sum_I C_I(\nu) \left[\xi_{00}^I - \xi_{01}^I \left(\ln\frac{M^2}{\nu^2} - \gamma_E\right) + \cdots\right]\right\} \\
b_+ &= \int_0^\infty \frac{dx}{x^2}\,\frac{\rho_+^{\pi\pi}(x)}{G_F} + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \frac{\pi^2}{6}\,\frac{M_K^2}{M^2} \sum_I C_I(\nu)\,\xi_{01}^I \\
&\quad + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times \lambda_+ \frac{M_K^2}{M_\pi^2} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \left\{ \frac{C_{7V}(\nu)}{\alpha} + \sum_I C_I(\nu) \left[\xi_{00}^I - \xi_{01}^I \left(\ln\frac{M^2}{\nu^2} - \gamma_E\right) + \cdots\right]\right\} \\
&\quad - \frac{1}{60}\left(\frac{M_K^2}{M_\pi^2}\right)^2 \left(\alpha_+ - \beta_+ \frac{s_0}{M_\pi^2}\right)
\end{aligned}
$$

$$
a_+ = -1.58 + \begin{cases} -0.07 \div +0.03 & \text{NDR} \\ -0.06 \div +0.01 & \text{HV} \end{cases}
$$

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b_+ = -0.76 + \begin{cases} -0.11 \div -0.01 & \text{NDR} \\ -0.11 \div -0.01 & \text{HV} \end{cases}
$$

Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

Putting everything together

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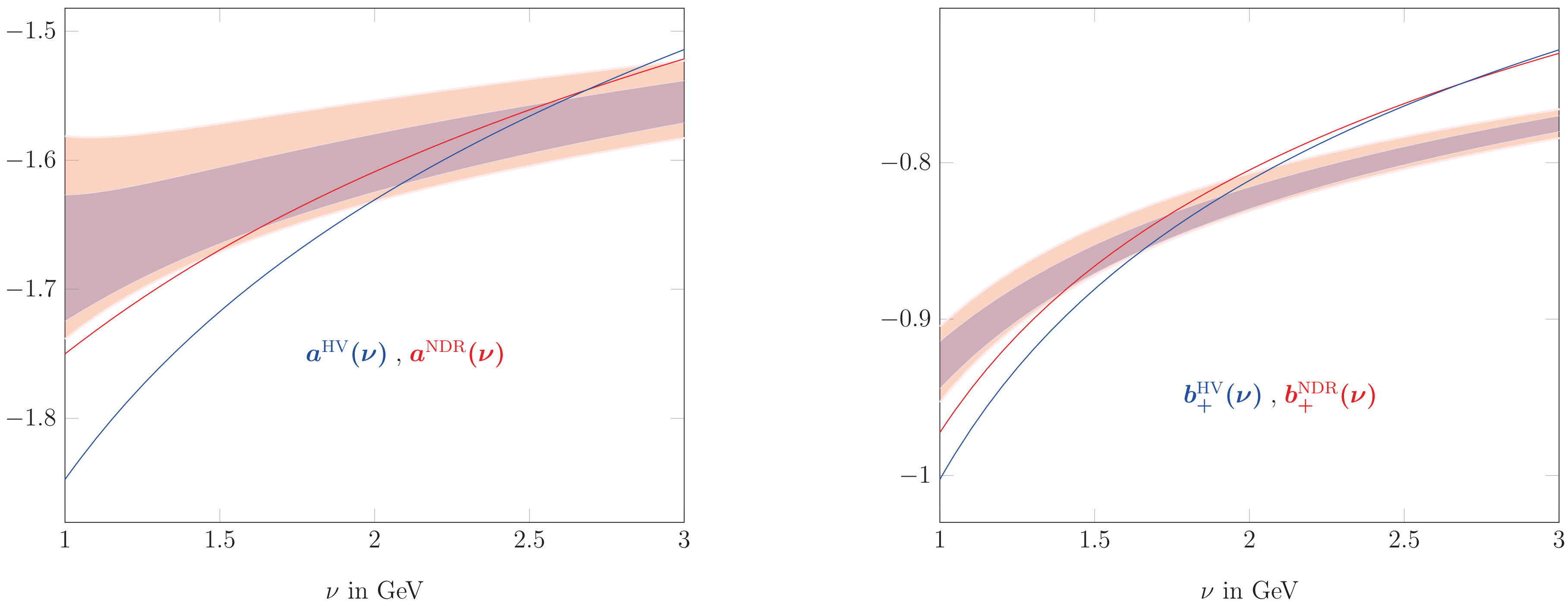
$$\begin{aligned} b_+ = & \int_0^\infty \frac{dx}{x^2}\,\frac{\rho_+^{\pi\pi}(x)}{G_F} + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \frac{\pi^2}{6} \frac{M_K^2}{M^2} \sum_I C_I(\nu)\, \xi_{01}^I \\ & + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times \lambda_+ \frac{M_K^2}{M_\pi^2} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \left\{ \frac{C_{7V}(\nu)}{\alpha} + \sum_I C_I(\nu) \left[\xi_{00}^I - \xi_{01}^I \left(\ln \frac{M^2}{\nu^2} - \gamma_E \right) \right] \right\} \\ & - \frac{1}{60} \left(\frac{M_K^2}{M_\pi^2} \right)^2 \left(\alpha_+ - \beta_+ \frac{s_0}{M_\pi^2} \right) \end{aligned}$$



Predicting $a_+,\ b_+$? Going beyond the low-energy expansion

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b_+ &= \int_0^\infty \frac{dx}{x^2}\,\frac{\rho_+^{\pi\pi}(x)}{G_F} + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \frac{\pi^2}{6}\,\frac{M_K^2}{M^2} \sum_I C_I(\nu)\, \xi_{01}^I \\
&\quad + \frac{f_+^{K^\pm\pi^\mp}(0)}{4\pi} \times \lambda_+ \frac{M_K^2}{M_\pi^2} \times 16\pi^2 \left(\frac{1}{\sqrt{2}} V_{us}^* V_{ud}\right) \left\{ \frac{C_{7V}(\nu)}{\alpha} + \sum_I C_I(\nu) \left[\xi_{00}^I - \xi_{01}^I \left(\ln \frac{M^2}{\nu^2} - \gamma_E \right) \right] \right\} \\
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$$
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$$

Difficult to assess theoretical errors at this stage

Values come in the right ballpark (encouraging)...

... for $\beta_+ \cdot 10^{-8} = -0.85$!

IV. Summary - Conclusions

- $K^\pm \to \pi^\pm \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$ offer a window to BSM physics (as, in general, other rare kaon decays)

- Unfortunately, they are long-distance dominated

- Amplitudes can be described by two parameters, $a_{+,S}$ and $b_{+,S}$ (assuming $\alpha_{+,S}$ and $\beta_{+,S}$ known)

- Rather precise data on decay distribution available for $K^\pm \to \pi^\pm \ell^+ \ell^-$

- Can expect more in the future (NA62), also for $K_S \to \pi^0 \ell^+ \ell^-$ (LHCb)

E. Goudzovski, https://indico.cern.ch/event/648004/contributions/2987967

R. Aaij et al. (LHCb coll.), arXiv:1808.08865

A. A. Alves Junior et al., arXiv:1808.03477

- Determinations of $a_{+,S}$ and $b_{+,S}$ require to go beyond the low-energy expansion

- Proposed structure of the form factor

$$W_+(z) = W_+^{\pi\pi}(z) + W_+^{\rm res}(z;\nu) + W_+^{\rm SD}(z;\nu)$$

Contribution from the two-pion state

- Phenomenological evaluation from unsubtracted dispersion relation

- Absorptive part provided by the e.m. form factor of the pion and by the P-wave projection of the $K^{\pm}\pi^{\mp} \to \pi^+\pi^-$ amplitude

- Simple approach (IAM unitarization) (β_+?)

- More elaborate tools (e.g. Khuri-Treiman eqs.) available

V. Bernard, S. Descotes-Genon, M. K., B. Moussallam, in progress

- Room for improvement in the future ($\bar{K}K$?, $K\pi$?...)

$$\longrightarrow \qquad W_{+S}(z) = W^{\pi\pi}_{+,S}(z) + W^{K\bar{K}}_{+,S}(z) + W^{\rm res}_{+,S}(z;\nu) + W^{\rm SD}_{+,S}(z;\nu)$$

Matching with the short-distance regime

- Form factors behave, in the asymptotic euclidian region, as $\sim \ln^p(-s/\nu^2)$, where ν is the renormalization scale

- Renormalization of SD singularity implemented through the operator Q_{7V} and its Wilson coefficient $C_{7V}(\nu)$

- SD behaviour results from the pile-up of more and more complex intermediate states

- This process has been described through a, necessary infinite, set of zero-width resonances intermediate states

- Possible to adjust the couplings of these resonances such as to reproduce the correct high-energy behaviour al LO and at NLO

Lepton flavour universality in the kaon sector?

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Prospects for phenomenological estimates of a_+ and b_+ look good, with some (reasonable) amount of additional theoretical work

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Complementary to existing and future efforts to address this issue through lattice QCD

Thank you for your attention