# A tentative to understand Jordan and Einstein Frames by Hamiltonian Analysis

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partially in collaboration with Matteo Galaverni and based on arXiv:2003.04304 Phys. Rev. D 103 024022 (2021) , arXiv:2110.12222 Phys. Rev. D 105 084008 (2022), arXiv:2112.02098 Universe 8 (2021), and arXiv:2310.09539 (submitted to PRD).

# Jordan-Einstein Frames

- Old paper: Dicke (Phys. Rev. (1962) **125,** 6 2163-2167)

Suppose the proton mass is $m_p$ in mass units $m_u$ and, in "natural units", we scale the unit of measurement by a factor $\lambda^{-1}$ (length)$^{-1}$ $\tilde{m}_u = \lambda^{-1} m_u$ . In the new unit the proton mass $\tilde{m}_p = \lambda^{-1} m_p$.

- Confronting the measurement of the proton mass in the two mass units (Faraoni and Nadeau 2007)

$$\frac{\tilde{m}_p}{\tilde{m}_u} = \frac{\lambda^{-1} m_p}{\lambda^{-1} m_u} = \frac{m_p}{m_u}$$

# Jordan-Einstein Frames

- Since $d\tilde{s} = \lambda ds$ and $ds = \left(g_{ij}dx^i dx^j\right)^{\frac{1}{2}}$, then the covariant metric functions scales as

$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$$

- Invariance under rescaling of units of measurement implies Weyl (conformal) invariance of the metric tensor

- The starting frame is called "Jordan" frame and the conformal transformed the "Einstein Frame". One observable can be computed in both frames. Its measure, obviously different in the two frames, is related by conformal rescaling according to the observable's dimensions.(e.g. $\tilde{m}_p = \lambda^{-1} m_p$ ).

# Scalar-Tensor Theory

- In general, one starts from a scalar-tensor theory, with GHY-like boundary term, in the Jordan Frame

$$S=\int_M d^n x\sqrt{-g}\left(f(\phi)R-\frac{1}{2}\lambda(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-U(\phi)\right)+2\int_{\partial M}d^{n-1}\sqrt{h}f(\phi)K$$

- and passes to the Einstein Frame with the transformation

$$\tilde{g}_{\mu\nu}=\Big(16\pi G f(\phi)\Big)^{\frac{2}{n-2}}g_{\mu\nu}\ ,$$

- therefore, the action becomes

$$S=\int_M d^n x\sqrt{-\tilde{g}}\left(\frac{1}{16\pi G}\tilde{R}-A(\phi)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right)+\frac{1}{8\pi G}\int_{\partial M}d^{n-1}\sqrt{\tilde{h}}\tilde{K}$$

$$A(\phi)=\frac{1}{16\pi G}\left(\frac{\lambda(\phi)}{2f(\phi)}+\frac{n-1}{n-2}\frac{(f'(\phi))^2}{f^2(\phi)}\right),V(\phi)=\frac{U(\phi)}{[16\pi G f(\phi)]^{\frac{n}{n-2}}}$$

- It is assumed that if $(g_{\mu\nu}(x),\phi(x))$ is solution of the E.O.M also $(\tilde{g}_{\mu\nu}(x,\phi),\phi(x))$ is solution (True?). This reasoning seems to address that the transformation from the Jordan to the Einstein frame look like a canonical transformation in the Hamiltonian theory.

# Brans-Dicke Theory

- Brans-Dicke, with GHY boundary term, is a particular case of Scalar Tensor theory ( $f(\phi) = \phi$ )

$$S = \int_M d^4x\sqrt{-g}\left(\phi\ {}^4R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K \quad .$$

Deruelle, Sendouda, Youssef PRD 80, (2009). They still claim that the transformations are Hamiltonian canonical

- How to perform canonical analysis of this theory?

Garay and Gracìa-Bellido NPB 400 (1993): the transformations are Hamiltonian canonical.

$$\bar{q}_{ik} = \Phi q_{ik}, \quad \phi = -\frac{1}{2\beta}\log\Phi$$

$$\bar{N}^2 = \Phi N^2, \quad \bar{N}_i = \Phi N_i$$

$$\tilde{h}_{ab} = \phi h_{ab}, \quad \tilde{N}^a = N^a, \quad \tilde{N} = \sqrt{\phi}N, \quad \tilde{\phi} = \sqrt{\frac{3}{2}}\ln\phi$$

$$\tilde{p}^{ab} = \frac{1}{\phi}p^{ab}, \quad \tilde{\pi} = \sqrt{\frac{2}{3}}(\phi\pi - p)$$

$$\{\tilde{h}_{ab}, \tilde{p}^{cd}\}_{\mathrm{J}} = \{h_{ab}, p^{cd}\}_{\mathrm{J}}, \quad \{\tilde{\phi}, \tilde{\pi}\}_{\mathrm{J}} = \{\phi, \pi\}_{\mathrm{J}}, \quad \{\tilde{p}^{ab}, \tilde{\pi}\}_{\mathrm{J}} = 0, \quad \{\tilde{h}_{ab}, \tilde{\phi}\}_{\mathrm{J}} = 0, \quad \{\tilde{h}_{ab}, \tilde{\pi}\}_{\mathrm{J}} = 0$$

$$\{\tilde{p}^{ab}, \tilde{\phi}\}_{\mathrm{J}} = 0$$

- N.B. ADM metric:

$$g = -(N^2 - N_iN^i)dt\otimes dt + N_i(dx^i\otimes dt + dt\otimes dx^i) + h_{ij}dx^i\otimes dx^j$$

# Brans-Dicke Theory

- The Hamiltonian Weyl (conformal) transformations from the Jordan to the Einstein frames are

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}}\,;\tilde{N}_i = N_i(16\pi G\phi)\,;\tilde{h}_{ij} = (16\pi G\phi)\,h_{ij}\,;\tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}}\,;$$

$$\tilde{\pi}^i = \frac{\pi^i}{(16\pi G\phi)},;\tilde{\pi}^{ij} = \frac{\pi^{ij}}{16\pi G\phi}\,;\phi = \phi\,;\tilde{\pi}_\phi = \frac{1}{\phi}(\phi\pi_\phi - \pi_h)$$

- They are not Hamiltonian canonical

$$\begin{aligned}
\{\widetilde{N}(x), \widetilde{\pi}_\phi(x')\} &= \frac{8\pi G N(x)\delta^{(3)}(x-x')}{\sqrt{16\pi G\phi(x)}} \neq 0\,, \{\widetilde{N}_i(x), \widetilde{\pi}_\phi(x')\} = 16\pi G N_i(x)\delta^{(3)}(x-x') \neq 0\,,\\
\{\widetilde{\pi}_N(x), \widetilde{\pi}_\phi(x')\} &= -\frac{8\pi G\pi_N(x)\delta^{(3)}(x-x')}{\sqrt{(16\pi G\phi(x))^3}} \neq 0\,, \{\widetilde{\pi}^i(x), \widetilde{\pi}_\phi(x')\} = -\frac{\pi^i(x)}{(16\pi G\phi^2)}\delta^{(3)}(x-x') \neq 0
\end{aligned}$$

- The Dirac's constraint analysis of the Hamiltonian theory has to be done, independently, in the Jordan and Einstein frames. We have studied the Hamiltonian constrained theory in Jordan and Einstein frames for both cases $\omega \neq -\frac{3}{2}$ and , $\omega = -\frac{3}{2}$. In the case $\omega = -\frac{3}{2}$ the theory has an extra Weyl(conformal) symmetry with an associated primary first class constraint $C_\phi$

| Hamiltonian Analysis of BD for $\omega \neq -\frac{3}{2}$ | |
|---|---|
| **in Jordan Frame** | **in Einstein Frame** |
| *constraints*<br>$\pi \approx 0; \pi^i \approx 0; \mathcal{H} \approx 0; \mathcal{H}_i \approx 0;$ | *constraints*<br>$\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{\mathcal{H}} \approx 0; \widetilde{\mathcal{H}}_i \approx 0;$ |
| *constraint algebra*<br>$\{\pi, \pi_i\} = 0; \{\pi, \mathcal{H}\} = 0; \{\pi, \mathcal{H}_i\} = 0; \{\pi_i, \mathcal{H}\} = 0;$<br>$\{\pi_i, \mathcal{H}_j\} = 0;\ \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial_i' \delta(x, x');$<br>$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j \delta(x, x') - \mathcal{H}_j(x)\partial_i{}' \delta(x, x');$<br>$\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i \delta(x, x') - \mathcal{H}^i(x')\partial_i' \delta(x, x');$ | *constraint algebra*<br>$\{\widetilde{\pi}, \widetilde{\pi}_i\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i\} = 0; \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}\} = 0;$<br>$\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j\} = 0;\ \left\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}_i(x')\right\} = -\widetilde{\mathcal{H}}(x')\partial_i' \delta(x, x');$<br>$\{\widetilde{\mathcal{H}}_i(x), \widetilde{\mathcal{H}}_j(x')\} = \widetilde{\mathcal{H}}_i(x')\partial_j \delta(x, x') - \widetilde{\mathcal{H}}_i(x)\partial_i{}' \delta(x, x');$<br>$\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}(x')\} = \widetilde{\mathcal{H}}^i(x)\partial_i \delta(x, x') - \widetilde{\mathcal{H}}^i(x')\partial_i' \delta(x, x');$ |

# BRANS-DICKE PARTICULAR CASE $\boldsymbol{\omega = -\frac{3}{2}}$

- The BD action for $\omega = -\frac{3}{2}$ is (for consistency reasons here U($\phi$)=$\alpha\phi^2$ $\alpha$ is a constant)

$$S^{(-3/2)} = \int_M d^4x\sqrt{-g}\left(\phi R + \frac{3}{2}\frac{g^{\mu\nu}}{\phi}\partial_\mu\phi\partial_\nu\phi - \alpha\phi^2\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K\ .$$

- It is invariant under this conformal transformations

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \widetilde{\phi} = \frac{\phi}{\Omega^2}$$

- that is

$$S^{(-3/2)} = \int_M d^4x\sqrt{-\widetilde{g}}\left(\widetilde{\phi}\widetilde{R} + \frac{3}{2}\frac{\widetilde{g}^{\mu\nu}}{\widetilde{\phi}}\partial_\mu\widetilde{\phi}\partial_\nu\widetilde{\phi} - \alpha\widetilde{\phi}^2\right) + 2\int_{\partial M} d^3x\sqrt{\widetilde{h}}\widetilde{\phi}\widetilde{K}\ .$$

# BRANS-DICKE PARTICULAR CASE $\boldsymbol{\omega = -\frac{3}{2}}$

- Clearly the Hamiltonian and momenta constraints are

$$\mathcal{H}^{(-3/2)} = \sqrt{h}\left\{ \left[-\phi\,^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right] - \quad \frac{3}{2\phi}D_i\phi D^i\phi + 2D^iD_i\phi + U(\phi)\right\}$$

$$\mathcal{H}_i^{(-3/2)} = -2D_j\pi_i^j + D_i\phi\pi_\phi$$

- We also have a further primary constraint due to conformal invariance

$$C_\phi \equiv \pi_h - \phi\pi_\phi \approx 0$$

- All the constraints (shown through lengthy and technically complicated calculations) are first class .

<table>
<tr><th colspan="2">Hamiltonian Analysis of BD for $\omega = -\frac{3}{2}$</th></tr>
<tr><th>in Jordan Frame</th><th>in Einstein Frame</th></tr>
<tr><td>constraints<br>$\pi_N \approx 0; \pi^i \approx 0; C_\phi \approx 0; \mathcal{H}^{(-3/2)} \approx 0; \mathcal{H}_i^{(-3/2)} \approx 0;$</td><td>constraints<br>$\widetilde{\pi}_N \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{C}_\phi = -\widetilde{\phi}\widetilde{\pi}_\phi \approx 0; \widetilde{\mathcal{H}}^{(-3/2)} \approx 0; \widetilde{\mathcal{H}}_i^{(-3/2)} \approx 0;$</td></tr>
<tr><td>constraint algebra<br>$\{\pi_N, \pi_i\} = \{\pi_N, \mathcal{H}^{(-3/2)}\} = \{\pi_N, \mathcal{H}_i^{(-3/2)}\} = 0;$<br>$\{\pi_i, \mathcal{H}^{(-3/2)}\} = \{\pi_i, \mathcal{H}_j^{(-3/2)}\} = 0;$<br>$\left\{C_\phi(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\partial_i'\delta(x,x')C_\phi(x');$<br>$\left\{C_\phi(x), \mathcal{H}^{(-3/2)}(x')\right\} = \frac{1}{2}\mathcal{H}^{(-3/2)}(x)\delta(x,x');$<br>$\left\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\mathcal{H}^{(-3/2)}(x')\partial_i'\delta(x,x');$<br>$\left\{\mathcal{H}_i^{(-3/2)}(x), \mathcal{H}_j^{(-3/2)}(x')\right\} = \mathcal{H}_i^{(-3/2)}(x')\partial_j\delta(x,x')$<br>$-\mathcal{H}_j^{(-3/2)}(x)\partial_i{}'\delta(x,x');$<br>$\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}^{(-3/2)}(x')\} =$<br>$\mathcal{H}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \mathcal{H}_i^{(-3/2)}(x')\partial'^i\delta(x,x') +$<br>$\left[D^i(\log\phi(x))\right] C_\phi(x)\partial_i\delta(x,x')$<br>$-\left[D^i(\log\phi(x'))\right] C_\phi(x')\partial_i'\delta(x,x');$</td><td>constraint algebra<br>$\{\widetilde{\pi}_N, \widetilde{\pi}_i\} = \{\widetilde{\pi}_N, \widetilde{\mathcal{H}}^{(-3/2)}\} = 0; \{\widetilde{\pi}_N, \widetilde{\mathcal{H}}_i^{(-3/2)}\} = 0;$<br>$\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}^{(-3/2)}\} = \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j^{(-3/2)}\} = 0;$<br>$\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = 0;$<br>$\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\right\} = 0;$<br>$\left\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = -\widetilde{\mathcal{H}}^{(-3/2)}(x')\partial_i'\delta(x,x');$<br>$\{\widetilde{\mathcal{H}}_i^{(-3/2)}(x), \widetilde{\mathcal{H}}_j^{(-3/2)}(x')\} = \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_j\delta(x,x')$<br>$-\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial_i{}'\delta(x,x');$<br>$\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\} =$<br>$\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_i'\delta(x,x');$</td></tr>
</table>

# FLAT FLRW Brans-Dicke theory

$$ds^2 = -N^2(t)dt^2 + a^2(t)dx^3$$

$$\mathcal{L}_{FLRW} = -\frac{6a\dot{a}^2}{N}\phi - \frac{6a^2\dot{a}}{N}\dot{\phi} + \frac{\omega a^3}{N\phi}(\dot{\phi})^2 - Na^3U(\phi)$$

$$\begin{tikzcd} H_T \arrow[r, "\text{JF -------> EF}"] \arrow[d] & \widetilde{H}_T \arrow[d] \\ \text{e.o.m} & \text{e.o.m} \arrow[l, "\text{JF <--------EF}"] \end{tikzcd}$$

.

$$\dot{N} \approx \lambda_N \,, \tag{1}$$

$$\dot{\pi}_N = -H \approx 0 \,, \tag{2}$$

$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right) \,, \tag{3}$$

$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi) \,, \tag{4}$$

$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right) \,, \tag{5}$$

$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} \tag{6}$$

$$\dot{N} \approx \frac{\widetilde{\lambda}_N}{(16\pi G\phi)^{\frac{1}{2}}} - \frac{N^2}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right) \,, \tag{1}$$

$$\dot{\pi}_N \approx -H + \frac{N\pi_N}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right) \,, \tag{2}$$

$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right) \,, \tag{3}$$

$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi) \,, \tag{4}$$

$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right) \,, \tag{5}$$

$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} + \frac{H}{2\phi} \,. \tag{6}$$

# LOOKING FOR CANONICAL EQUIVALENCE

- Gauge fixing of the lapse N implemented as secondary constraints

$$\chi_0 \equiv N - c_0 \approx 0 \;\; ; \;\; \chi_1 \equiv \pi_N \approx 0 \quad \text{become second class constraints in the JF}$$

$$\widetilde{\chi}_0 \equiv \widetilde{N} - c_0\,(16\pi G\phi)^{\frac{1}{2}} \approx 0 \;\; ; \;\; \widetilde{\chi}_1 \equiv \widetilde{\pi}_N \approx 0 \;\; \text{are also second class constraints in EF}$$

- We define Dirac's Brackets (in the Jordan and Einstein frames)

$$\{\cdot\,,\,\cdot\}_{DB} \equiv \{\cdot\,,\,\cdot\} - \{\cdot\,,\chi_\alpha\}\,C^{-1}_{\alpha\beta}\,\{\chi_\beta,\,\cdot\} \qquad\qquad C_{\alpha\beta} \equiv \{\chi_\alpha, \chi_\beta\}$$

- Using Dirac's brackets, the dynamics stays on the manifold defined by second class constraints

$$\dot{\chi}_0 \approx \{N - c_0, H_T\}_{DB} \approx 0 \qquad\qquad \dot{\widetilde{\chi}}_0 \approx \left\{\widetilde{N} - c_0(16\pi G\phi)^{\frac{1}{2}}, \widetilde{H}'_T\right\}_{DB} \approx 0$$

$$\dot{\chi}_1 \approx \{\pi_N, H_T\}_{DB} \approx 0 \qquad\qquad \dot{\widetilde{\chi}}_1 \approx \left\{\widetilde{\pi}_N, \widetilde{H}'_T\right\}_{DB} \approx 0$$

# E.O.M IN J.F. AND E.F. WITH DIRAC'S BRACKETS

- We derive the equations of motions using Dirac's brackets, then we impose strongly the second-class constraints both in Jordan and Einstein frames

$$\dot{a} \approx -\frac{c_0}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi}+\frac{\pi_\phi}{a}\right), \tag{1}$$

$$\dot{\pi}_a \approx -\frac{c_0}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi}+\frac{2\pi_a\pi_\phi}{a}-\frac{3\phi\pi_\phi^2}{a^2}\right) -3c_0a^2U(\phi), \tag{2}$$

$$\dot{\phi} \approx \frac{c_0}{2a^2(2\omega+3)}\left(-\pi_a+\frac{2\phi\pi_\phi}{a}\right), \tag{3}$$

$$\dot{\pi}_\phi \approx -\frac{c_0}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2}+\frac{\pi_\phi^2}{a^2}\right)-c_0a^3\frac{dU}{d\phi}. \tag{4}$$

$$\dot{\widetilde{a}} \approx -c_0(16\pi G\phi)^{\frac{1}{2}}\frac{4\pi G\widetilde{\pi}_a}{3\widetilde{a}}, \tag{1}$$

$$\dot{\widetilde{\pi}}_a \approx c_0(16\pi G\phi)^{\frac{1}{2}}\left[-\frac{(2\pi G)\widetilde{\pi}_a^2}{3\widetilde{a}^2}+\frac{3(8\pi G)\widetilde{\pi}_\phi^2\widetilde{\phi}^2}{(2\omega+3)\widetilde{a}^4} -3\widetilde{a}^2V(\phi)\right], \tag{2}$$

$$\dot{\phi} \approx c_0(16\pi G\phi)^{\frac{1}{2}}\frac{(16\pi G)\widetilde{\pi}_\phi\phi^2}{(2\omega+3)\widetilde{a}^3}, \tag{3}$$

$$\dot{\widetilde{\pi}}_\phi \approx -c_0(16\pi G\phi)^{\frac{1}{2}}\left[\frac{16\pi G\widetilde{\pi}_\phi^2\widetilde{\phi}}{(2\omega+3)\widetilde{a}^3}+\widetilde{a}^3\frac{dV(\phi)}{d\phi}\right]. \tag{4}$$

- The transformation from Jordan to Einstein frame, having eliminated the Lapse N and its conjugate momenta $\pi_N$, is Hamiltonian canonical. The JF equations of motion are equivalent to the EF equations of motion.

## CANONICAL EQUIVALENCE OF JF AND EF VIA GAUGE FIXING

- Following the flat FLRW case, we gauge-fix the lapse N and the shifts $N_i$ in the JF and EF

$$\chi_0 \equiv N - c_0 \approx 0\ ,\ \chi_i \equiv N_i - c_i \approx 0\ ,\qquad \widetilde{\chi}_0 \equiv \widetilde{N} - c_0\,(16\pi G\phi)^{\frac{1}{2}}\ ,\ \widetilde{\chi}_i \equiv \widetilde{N}_i - c_i\,(16\pi G\phi)\ ,$$
$$\chi_4 \equiv \pi_N \approx 0\ ,\ \chi_{i+4} \equiv \pi_i \approx 0 \qquad \widetilde{\chi}_4 \equiv \widetilde{\pi}_N\ ,\ \widetilde{\chi}_{i+4} \equiv \widetilde{\pi}^i$$

- These constraints are second class. Then, we define suitable Dirac's brackets, verify that the Hamiltonian and momentum constraints stay first class.

- The dynamics remains on the second-class constraint manifold both in JF and EF

$$\dot{N} \approx \{N, H_T\}_{DB} \approx 0 \quad , \quad \dot{N}_i \approx \{N_i, H_T\}_{DB} \approx 0 \qquad \dot{\widetilde{\chi}}_0 \approx \{\widetilde{\chi}_0, \widetilde{H}'_T\}_{DB} \approx 0\, ,\ \dot{\widetilde{\chi}}_i \approx \left\{\dot{\widetilde{\chi}}_i, \widetilde{H}'_T\right\}_{DB} \approx 0\ ,$$
$$\dot{\pi}_N \approx \{\pi_N, H_T\}_{DB} \approx 0 \quad , \quad \dot{\pi}^i \approx \{\pi^i, H_T\}_{DB} \approx 0 \qquad \dot{\widetilde{\pi}}_N \approx \{\pi_N, \widetilde{H}'_T\}_{DB} \approx 0\, ,\ \dot{\widetilde{\pi}}^i \approx \{\widetilde{\pi}^i, \widetilde{H}'_T\}_{DB} \approx 0$$

## CANONICAL EQUIVALENCE OF JF AND EF VIA GAUGE FIXING

- The evolution of the other variables are calculated implementing the Dirac's bracket (Db)

- Once calculated the e.o.m. by the Db, we solve the second-class constraints and substitute them in the e.o.m.

- On this reduced phase space, without the lapse and the shifts variables, the transformation from the Jordan to the Einstein frame is Hamiltonian canonical.

- The equations of motion are completely equivalent , on the reduced phase space in the two frames.

- Does it mean that JF and EF are physically equivalent?

# CANONICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

- Harmonic Oscillator (Goldstein )

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

- Canonical transformations (not symmetry of the system…)

$$q = \sqrt{\frac{2P}{m\omega}} sinQ \, , p = \sqrt{2m\omega P} cosQ$$

- Therefore the Hamiltonian becomes

$$H = \omega P$$

- and then,

$$P = \frac{E}{\omega}, \;\; \dot{Q} = \frac{\partial H}{\partial P} = \omega, \;\; Q = \omega t + \alpha, \;\; q(t) = \sqrt{\frac{2E}{m\omega^2}} sin(\omega t + \alpha)$$

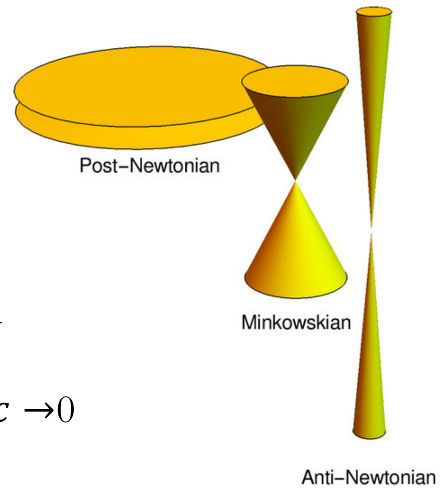
- Notice that the harmonic oscillator is mapped into a free particle

# ANTI-GRAVITY TRANSFORMATIONS (Canonical Transformations)

- There exist Hamiltonian Canonical Transformations on the extended phase space: The Anti-Gravity transformations

$$\widetilde{N}^{*} = N \; ; \; \widetilde{\pi}_{N^{*}} = \pi_{N} \; ; \widetilde{N_{i}^{*}} = N_{i} \; ; \; \widetilde{\pi}^{*i} = \pi^{i} \; ; \; \widetilde{h}^{*}_{ij} = (16\pi G\phi) h_{ij} \; ;$$

$$\widetilde{\pi}^{*ij} = \frac{\pi^{ij}}{(16\pi G\phi)^{\frac{1}{2}}} \; ; \widetilde{\phi}^{*} = \phi \; ; \; \widetilde{\pi}^{*}_{\phi} = \frac{1}{\phi}\left(\phi\pi_{\phi} - \pi_{h}\right);$$




Carrolian Gravity, G→ ∞ , $c$ →0

M. Niedermaier 2019

- In two dimensions, they look like

$$ds^2 = -dt^2 + \lambda^2 dx^2; \lambda > 1$$

- Since this theory is canonically equivalent to B-D theory, the constraint algebra of secondary first class constraints ($\mathcal{H}, \mathcal{H}_i$) is like B-D theory's one.

# Anti-Newtonian frame

- The ADM Hamiltonian in the Anti-Newtonian "frame " is

$$\mathcal{H}_{ADM} = \frac{\sqrt{\widetilde{h}}\widetilde{N^*}(\phi)^{\frac{1}{2}}}{(16\pi G)^{\frac{1}{2}}}\Bigg[ -{}^3\widetilde{R} + \frac{(16\pi G)^2}{\widetilde{h}}\left(\widetilde{\pi}^{ij}\widetilde{\pi}_{ij} - \frac{\widetilde{\pi}_h^2}{2}\right)$$

$$+\frac{(\omega+\frac{3}{2})}{\phi^2}\partial_i\phi\partial^i\phi \ + \frac{64(\pi G)^2\phi^2}{h(\omega+\frac{3}{2})}\widetilde{\pi}_\phi^2 + \widetilde{V}(\phi)\Bigg]$$

$$-2\widetilde{N^*}^i\widetilde{D}_j\widetilde{\pi}_i^j + \widetilde{N^*}^i\partial_i\phi\widetilde{\pi}_\phi$$

- Since this theory is canonically equivalent to BD theory, the constraint algebra of secondary first-class constraints $(\mathcal{H}, \mathcal{H}_i)$ is like that of BD theory.

- Is this theory, in the Anti-Newtonian frame, physically equivalent to BD theory?

# CANONICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

- BD-Theory in JF is canonical equivalent, via gauge-fixing of Lapse $N$ and shifts $N_i$ , to Einstein-GR, minimally coupled to a scalar field, in EF.

- JF-EF transformation preserves the light-cone structures (Weyl (conformal) transformation conserves the angles )

- BD-theory in JF is canonical equivalent to the "Anti-Newtonian" gravity, in the Anti-Newtonian frame. (light cone structure modified by Anti-Newtonian transformation).

- BD-theory cannot be equivalent to two physically inequivalent theories. Therefore, Hamiltonian canonical transformation represents, in our opinion, a mere mathematical equivalence. This transformation maps solutions of e.o.m in one frame into solutions of e.o.m in the other frame.

# CONCLUSIONS

- The transformations from the Jordan to the Einstein frames, in the extended phase space, are not Hamiltonian canonical transformations.

- Gauge-fixing the Lapse N and the Shifts $N_i$ and implementing the Dirac's Brackets, Hamiltonian canonical transformations do exist from JF to EF.

- This very fact does not mean, necessarily, that the two frames are "physically" equivalent.

- The equivalence of the physical observables in JF and EF remains still to be studied.

# Lemaître Conference 2024

## Black Holes, Gravitational Waves and Space-Time Singularities

Second international conference to celebrate the legacy of G. Lemaître

**Vatican Observatory**
**Castel Gandolfo, Italy**

**17 - 21 June 2024**

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Building upon the success of the inaugural Lemaître workshop we delve deeper into the legacy of Mgr. Georges Lemaître's profound insights. This iteration focuses on addressing paramount themes: Cosmology and the perplexing Hubble tension, the enigmatic nature of spacetime singularities encompassing the Big Bang and Black Holes, the Gravitational Waves they produce, the tantalizing pursuit of Quantum Gravity and its connections with the Entanglement and foundations of Quantum Theory.
The main goal of this workshop is to encourage interaction among the participants, between theory and observation, and to provide a stimulating and thought-provoking environment for new ideas.

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** to be confirmed*

Georges Lemaître


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