MY TRAJECTORY:

- LAUREA MAGISTRALE in PHYSICS
  PROF. FEDELE LIZZI, PROF. PATRIZIA VITALE
- PHD IN MATHEMATICS at UNIVERSITY OF COPENHAGEN
- POSTDOCS (SPAIN, GERMANY...)
- TENURE TRACK (RITA LEVI MONTALCINI) 2018
  back to Italy :)
- 2021 → ASSOCIATE PROF in GEOMETRY

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QUANTIZATION : MOTIVATION FROM PHYSICS

quantization is sort of process

CLASSICAL MECHANICS $\rightsquigarrow$ QUANTUM MECHANICS

in order to understand this process we first have to fix notation

What are $\underbrace{\text{OBSERVABLES}}$ in both theories

quantities you can measure in your physical system like momenta, energy ...

FUNCTIONS ON PHASE SPACE

certain class of, polynomial functions, smooth and so on

OPERATORS ON HILBERT SPACE

certain class of, bounded, self adjoint ...

QUANTIZATION : MOTIVATION FROM PHYSICS

what do we care about observables ? Their **ALGEBRAIC** structure

FUNCTIONS ON PHASE SPACE

OPERATORS ON HILBERT SPACE

pointwise

PRODUCT

concatenation

$\Downarrow$

ASSOCIATIVE

$*$ - **ALGEBRA**

involution

$*$ - algebra

usually we talk about $\mathbb{C}$ - valued functions

$*$ = complex conjugation

operators have adjoints

$*$ = operator adjoint

WE ARE IGNORING ALL DETAILS OF FUNCTIONAL ANALYSIS !

QUANTIZATION : MOTIVATION FROM PHYSICS

ONCE WE HAVE $\mathcal{A}$ IT'S CLEAR WHAT STATES ARE : Positive functionals on $\mathcal{A}$

FOCUS ON OBSERVABLE ALGEBRA! Typically it's very hard to make a passage from classical to quantum mechanics looking at states. How to associate a wave function to a point on phase space, how to cook up the Hilbert space in a way this correspondence works. Much easier for observables : energy $\mapsto$ energy, algebraic structures are easier to transfer and states will come automatically.

OBSERVABLE ALGEBRAS : what are the differences?

CLASSICAL OBSERVABLE ALGEBRA

phase space M smooth manifold

$+$ Poisson structure $\pi \in \Gamma^\infty(\Lambda^2 TM)$

(just a tensor, think locally $\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$)

$\Rightarrow$ $C^\infty(M)$ associative algebra COMMUTATIVE

$+$ $\underbrace{\text{Poisson}}_{\text{Lie + Leibniz}}$ bracket $\{f, g\} := \pi(df, dg)$

Why Poisson? it allows us to write Hamilton equations of motion

$$\frac{df(t)}{dt} = \{f(t), H\}$$ TIME EVOLUTION

Hamiltonian function

QUANTUM OBSERVABLE ALGEBRA

bounded operators $\mathcal{B}(\mathcal{H})$ associative algebra

NON COMMUTATIVE

$\Rightarrow$ we have an intrinsic bracket (Lie)

$$[A, B] = AB - BA$$

Again, important because it encodes time evolution

$$\frac{dA(t)}{dt} = \frac{1}{i\hbar}[A(t), H]$$

Hamiltonian operator

Planck constant

QUANTIZATION

Now we know better what we want

COMMUTATIVE $\longrightarrow$ NON COMMUTATIVE

POISSON BRACKET $\longrightarrow$ COMMUTATOR

we want to preserve $\underbrace{\text{as much}}_{\text{depends on the program}}$ algebraic features as possible

idea : a classical obs $\rightsquigarrow$ $\hat{a}$ quantum obs

we want this correspondence to be bijection } strong claim that turns out to be wrong

why ? quantum observable space bigger than classical ?
$\Rightarrow$ quantization is hopeless

classical space bigger ? weird, classical features not detectable in quantum space ... actually the contrary !

QUANTIZATION

keep in mind that we have made certain idealizations, smooth, bounded, not always the case in Lab! Maybe that's the reason for failing the bijection.

WISH THINKING

$$a \rightsquigarrow \hat{a}$$
$$1:1$$

LINEARITY $\qquad \widehat{za + wb} \rightsquigarrow z\hat{a} + w\hat{b}$

*-INVOLUTION IS PRESERVED $\qquad \hat{\bar{a}} \rightsquigarrow \hat{a}^*$

PRODUCTS PRESERVED $\qquad \widehat{ab} \rightsquigarrow \hat{a}\,\hat{b}$ $\equiv$ SPECTRA SHOULD CORRESPOND

↳ good for determine spectral values of algebra elements

BRACKET PRESERVED $\qquad \widehat{\{a, b\}} \rightsquigarrow \frac{1}{i\hbar}[\hat{a}, \hat{b}]$ $\equiv$ MOTION SHOULD CORRESPOND

DEFORMATION QUANTIZATION

PROBLEM : THIRD AND FOURTH CONTRADDICT !

WAY OUT : ABANDON ONE OF THEM

GEOMETRIC QUANTIZATION : abandon 3rd one (one gets problems with spectra)

DEFORMATION QUANTIZATION : relax all of them (but 1st)

$\rightsquigarrow$ means correspondence up to higher orders in $\hbar$

we have a correction that disappears in classical limit $\hbar \to 0$

REMARK

We have to introduce a continuous (smooth, analytic...) dependence on $\hbar$ such that classical limit makes sense

STAR PRODUCT

DEFINITION [BFFLS '78]

Let (M, π) be a Poisson manifold.

Then a FORMAL STAR PRODUCT is an associative $\mathbb{C}[\![\hbar]\!]$-bilinear product $\star$ for $C^\infty(M)[\![\hbar]\!]$

$$f \star g = \sum_{k=0}^{\infty} \hbar^k C_k(f,g)$$

such that

i) $C_0(f,g) = fg$

ii) $C_1(f,g) - C_1(g,f) = i\{f,g\}$

iii) $f \star 1 = 1 \star f = f$

iv) C_k bidifferential operators

QUANTIZATION IN THE HAND OF MATHEMATICIANS ...

... becomes a map from π to $\star$, meaning from a certain bivector field to certain bidifferential operators

FORMALITY THEOREM (Kontsevich '97)

$\exists$ L_∞-morphism between multivector fields and multidifferential operators.

literally : Lie algebra morphism up to homotopy, meaning with correction terms

REMEMBER THE PHILOSOPHY

$$\widehat{\{a, b\}} \rightsquigarrow \frac{1}{i\hbar} [\hat{a}, \hat{b}]$$

$\rightsquigarrow$ is = + corrections !

TIME FOR QUESTIONS

THANKS FOR

THE

ATTENTION

ADVANTAGES OF DEFORMATION QUANTIZATION

- its generality : other schemes use the specific features of the classical system , DQ is universal as its requirements are minimal : only Poisson manifold.
  The hard part is proving existence (and classification) of star products

- physical interpretation of observables fixed from beginning : they simply stay the same elements of the same underlying vector space, only the product law changes. It's clear what the Hamiltonian will be, the same as classical !

PRICE TO PAY...

- we asked $\star$ to be associative : hard, it results in a infinite chain of quadratic equations for C_k operators [formality theorem will solve this issue]

- main difficulty : the deformation parameter correspond to a fundamental constant of nature, NON zero and NOT dimensionless. It makes no sense to speak of "smallness of $\hbar$" : convergence of the series $\star$ becomes an important issue.